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TABLE OF CONTENTS

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Research Investigation of Laser Line Profiles

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SUMMARY

Some aspects of the theory of a gas laser recently developed by W. E. Lamb, Jr. are recast in a form which more fully displays the role played by the particle dynamics. The Wigner distribution function is used to derive kinetic equations which govern the external center of mass motion of the two-level systems as well as their internal dynamics. The effect of long range forces is discussed by treating the collision integral in a manner similar to that employed in plasma kinetic theory. A modification in the criterion for the existence of a dip in the output is obtained. It is also shown that effects due to long range forces are most noticeable at long optical wavelengths and when there is a large difference between the lifetimes of the two laser levels.

The experimental system for measuring the line profile of a dc excited argon laser has been designed, constructed, and made operative. Preliminary data on the "Lamb Dip" of an argon ion laser has been obtained as a function of pressure and excitation. The results of the experimental portion of this program and attempts to interpret the experiments in terms of this theory will be reported in later reports.

INTRODUCTION

The objective of this program is to conduct an experimental and theoretical investigation of laser line profiles. The theoretical analysis will be based on a discussion of many particle distribution functions which provide a single unified theory for treating both the long range nature of the coulomb forces and the quantum mechanical interaction of the two-level system with the optical field. The effects of such long range forces on the steady state conditions in a single mode gas laser will be determined. The experimental program will determine the homogeneous line width in an argon ion laser and relate experimental effects to the prediction from the theoretical portion of the investigation.

KINETIC THEORY FOR AN ION LASER

Introduction

A satisfactory description of the operating conditions for a gas laser including effects due to rectilinear particle motion is presently available.(1) A modification of this theory to include the effect of short range collisions has also been developed.(2) However, a theoretical framework permitting the systematic calculation of corrections associated with particle interactions has yet to be provided. The purpose of this work is to show that such refinements appear quite naturally when the dynamics of the laser medium is described in terms of the Wigner distribution function.

Following the technique which has been employed recently in both classical and quantum plasmas, the Liouville equation

for an N-particle Wigner distribution function is first obtained. This distribution function contains coordinates describing both the external center of mass motion of each particle (and is treated classically) as well as the internal motion which is quantized. From the Liouville equation, an equation for a one particle distribution function is obtained which contains an integral over a two particle function. This integral contains all effects associated with particle interactions and is exact. To treat the long range coulomb forces that are present in the ionized gas laser, the collision integral is treated by methods developed recently in classical plasma physics. Implicit in this treatment is the standard assumption of plasma physics that one is dealing with a completely ionized system so that short range collisions among neutral atoms may be neglected. Unfortunately, this idealization is not realizable in practice. The effect of long range collisions on the steady state conditions of an ion laser operating in a single mode is then derived from a third order perturbation analysis.

The N-Particle Wigner Distribution Function

We consider a cavity of volume V containing N two-level systems interacting with an electromagnetic field. Each particle is assumed to have a mass m and the velocity distributions of particles in each of the two levels are assumed to be Maxwellian at the same temperature T. The i^{th} such system has internal coordinates p_i, q_i, and external center of mass coordinates P_i, Q_i.

The Hamiltonian operator governing the dynamics of the N two-level systems is assumed to be composed of four types of energy operators. They are the kinetic energy operator K_i for the center of mass motion of the i^{th} particle, H_i the operator associated with the internal energy of an isolated two-level system, V_i the

the potential energy of a particle interacting with the electromagnetic field and finally, V_{ij} the two-body potential expressing the interaction between pairs of two-level systems in the cavity. In actual laser systems, the scattering will be predominantly with particles other than those in either of the two laser levels. Hence, to describe the scattering, it will be necessary to consider a three-level system. Since optical transitions to this third level will be ignored, only a minor extension of the two-level analysis will be required.

The Hamiltonian is, then,

$$\mathcal{H} = \sum_{i=1}^{N} (K_i + H_i + V_i) + \sum_{i,j} V_{ij} \quad . \tag{1}$$

The two-body potential affects the internal dynamics of the two-level systems as well as their translational motion. It will be assumed that these two effects are additive so that one may decompose the potential into the form

$$\sum_{i,j} V_{ij} = \sum_{i<j} V_{ij}\,(|Q_i - Q_j|) + \sum_{i,j} W_{ij}\,(q_i, |Q_i - Q_j|) \ , \tag{2}$$

where U_{ij} is the two-body potential influencing the center of mass motion and W_{ij} represents the effect of the external field of the j^{th} particle upon the internal dynamics of the i^{th} particle.

The first and third types of terms in the Hamiltonian are, respectively,

$$K_i = -\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial Q_i^2} \tag{3}$$

$$V_i = eE(Q_i, t)\cdot q_i \tag{4}$$

The operator H_i and the second term in Eq. (2) are defined in terms of the Schroedinger equation

$$(H_i + \sum_j W_{ij})\,\psi_i(q_i, t) = i\hbar\,\frac{\partial}{\partial t}\,\psi(q_i, t) \ . \tag{5}$$

The spatial dependence of W_{ij} may be replaced by a stochastic time dependence. Furthermore, since the time required for a collision to take place (10^{-13} sec) is much longer than the period of the optical field (10^{-15} sec), Eq. (5) may be solved in the adiabatic approximation yielding wave functions of the form

$$\psi_{in}(q_i,t) = \chi_{in}(q_i)\exp\left[-i\tilde{E}_n t/\hbar - (i/\hbar)\int_{-\infty}^{t} dt' \Delta E_n(t')\right], \quad n = a,b \quad , \tag{6}$$

where

$$\Delta E_n(t) = \Delta E_n(Q_1,\ldots Q_N) = \sum_j W_{ij} \quad , \tag{7}$$

and the χ_{in} and $\tilde{E}_n$ are defined by the eigenvalue equation

$$H_i \chi_{in} = \tilde{E}_n \chi_{in} \quad . \tag{8}$$

The consta ts a and b refer, respectively, to the upper and lower levels of an individual two-level system. It is assumed that the $\tilde{E}_n$ contain an imaginary part which accounts for decay to the ground state labelled by the subscript c, i.e.,

$$\tilde{E}_n = E_n - i\hbar\gamma_n/2 \; . \tag{9}$$

As a result of this damping, the number of particles in states a and b decreases with increasing time. It will be assumed that these excited states are replenished by an external pumping mechanism to be introduced later.

Associated with the total energy operator $\mathcal{H}$ is an N-particle wave function satisfying the Schroedinger equation

$$\mathcal{H}\Psi(\eta,t) = i\hbar\frac{\partial\Psi}{\partial t} \quad , \tag{10}$$

where η represents the set of all position coordinates $\eta = (Q,q) = (Q_1,\ldots Q_N,\ q_1,\ldots q_N)$. The N-particle Wigner distribution function may now be introduced through the definition

$$f_N(\xi,\eta,t) = \int \frac{d^{6N}\lambda}{(2\pi)^{6N}} e^{-i\lambda\cdot\xi}\, \Psi^*(\eta - \frac{\hbar}{2}\lambda, t)\Psi(\eta + \frac{\hbar}{2}\lambda, t) \, , \tag{11}$$

where ξ represents the set of all momentum coordinates $\xi = (P,p) = (P_1,\ldots P_N,\ p_1,\ldots p_N)$, and the 6N dimensional transform variable λ has components $(T_1,\ldots T_N,\ \tau_1,\ldots\ \tau_N)$ corresponding to the 3N external and 3N internal coordinates respectively.

The Polarization of the Medium and Reduced Distribution Functions

As a result of using the Wigner distribution function, quantum statistical averages are obtained from expressions closely resembling those used in classical statistical mechanics. In particular, the polarization density of the medium due to the i^{th} two-level system is

$$\mathcal{P}_i(Q_i,t) = \int d^{3(N-1)}Q \int d^{3N}p \int d^{3N}q \int d^{3N}p\,(-eq_i)\,f_n\,(\xi,\eta,t) \tag{12}$$

$$= -\frac{e}{V}\int d^3P_i\,d^3p_i\,d^3q_i\,q_i\,f_1(\xi_i,\eta_i,t)\ , \tag{13}$$

where f_1 is the reduced distribution function obtained by integrating f_N over all phase space except that of the i^{th} particle, i.e.,

$$f_1(\xi_i,\eta_i,t) \equiv V\int d^{6(N-1)}\xi\,d^{6(N-1)}\eta\,f_N(\xi,\eta,t)\ . \tag{14}$$

For later use, a two-particle reduced distribution function for a pair of particles i and j is defined in a similar way as

$$f_2(\xi_i,\xi_j,\eta_i,\eta_j,t) \equiv V^2\int d^{6(N-2)}\xi\,d^{6(N-2)}\eta\,f_N(\xi,\eta,t)\ . \tag{15}$$

The one-particle Wigner distribution function also has a representation in terms of one-particle wave functions as

$$f_1(\xi_i,\eta_i,t) = V\int \frac{d^6\lambda}{(2\pi)^6}\,e^{-i\lambda\cdot\xi_i}\,\psi_i^*(\eta_i - \frac{\hbar}{2}\lambda_i,t)\,\psi(\eta_i + \frac{\hbar}{2}\lambda_i,t)\ , \tag{16}$$

where

$$\psi_1(\eta_1, t) = \sum_{\nu,n} a_{\nu n}(t)\phi_\nu(Q_1, t)\chi_n(q_1). \tag{17}$$

The sum on n extends over a, b and c while that on v is actually over the continuous values of translational momentum. With the expansion given in Eq. (17), the one-particle Wigner distribution function may be written

$$f_1(\xi_1, \eta_1, t) = \sum_{m,n} F_{1nm}(P_1, Q_1, t)\, f_{1mn}(p_1, q_1, t)\ , \quad m,n = a,b,c \tag{18}$$

where

$$F_{1nm}(P_1, Q_1, t) = V\sum_{\mu,\nu} a^*_{\mu m}(t)\, a_{\nu n}(t)\int \frac{d^3T_1}{(2\pi)^3} e^{-iT_1\cdot P_1}\phi^*_\mu(Q_1 - \frac{\hbar}{2}T_1, t)\,\phi_\nu(Q_1 + \frac{\hbar}{2}T_1, t) \tag{19}$$

$$f_{1mn}(p_1, q_1, t) = \int \frac{d^3\tau_1}{(2\pi)^3} e^{-i\tau_1\cdot p_1}\chi^*_m(q_1 - \frac{\hbar}{2}\tau)\,\chi_n(q_1 + \frac{\hbar}{2}\tau) \quad . \tag{20}$$

From these definitions it follows that

$$\begin{aligned} F_{1nm} &= F^*_{1mn} \\ f_{1nm} &= f^*_{1mn} \end{aligned} \quad . \tag{21}$$

Now $\mathcal{P}_i(Q_i, t)$ is the polarization density due to the i^{th} particle at point Q_i in the cavity. The polarization density at some point Q in the cavity due to the entire assemblage of N two-level systems is

$$\mathcal{P}(Q,t) = \sum_{i=1}^{N} \mathcal{P}_i(Q_i, t)\Big|_{Q_i = Q} = -n^{(0)}e\int d^3P\, d^3p\, d^3q\, q\, f_1(P,Q,p,q,t)\ , \tag{22}$$

where $n^{(0)}$ is the sum of the average number density of particles in all levels, i.e., $n^{(0)} = n_a^{(0)} + n_b^{(0)} + n_c^{(0)}$.

With the form of f_1 given in Eq. (18), the macroscopic

polarization density becomes

$$\mathcal{P}(Q,t) = -n^{(0)}\left[\wp_{ba}\int d^3P F_{lab}(P,Q,t) + \wp_{ca}\int d^3P F_{lab}(P,Q,t) + \wp_{cb}\int d^3P F_{lab}(P,Q,t)\right] + \text{c.c.}, \quad (23)$$

where

$$\wp_{mn} = e\int d^3q\, \chi_m^*(q)\, q\, \chi_n(q) = \wp_{nm}^* . \quad (24)$$

In obtaining Eq. (23) it has been assumed that the off diagonal matrix elements are real and hence equal. Also, it is assumed that there are no permanent dipole moments so that the diagonal matrix elements vanish.

The optical field and the associated polarization of the medium may be expanded in terms of the eigenmodes of the cavity $U_n(Q)$ which satisfy a vector Helmholtz equation.(1) One then has

$$\begin{aligned} E(Q,t) &= \sum_n E_n(t)\, U_n(Q) \\ \mathcal{P}(Q,t) &= \sum_n \mathcal{P}_n(t)\, U_n(Q) . \end{aligned} \quad (25)$$

Assuming that the optical field in the cavity varies in only one direction (labeled by a Z axis) and that the length of the cavity in this direction is L, a set of eigenfunctions adequate for the present work are

$$U_n(Q) = e \sin k_n Z , \quad (26)$$

where e is a unit vector perpendicular to the Z axis,

$$k_n = n\pi/L , \quad (27)$$

and n is a large positive integer. It will also be assumed that the polarization of the individual two-level system takes place along e so the $q = e|q|$ and $\wp_{mn} = e|\wp_{mn}|$. Finally, since we are only concerned with the contribution of level c to the particle dynamics and not to laser action between levels a and c or b and c,

the dipole moments φ_{ac} and φ_{bc} will be set equal to zero.

From Eqs. (23) and (25) we then have

$$\mathcal{P}_n(t) = -\mathcal{E}_n^{(0)} \frac{2}{L} \int_0^L dZ\, U_n(Z) \int d^3P\, F_{1ab}(P,Q,t) + \text{c.c.}\,, \qquad (28)$$

where $\varphi = \varphi_{ab} = \varphi_{ba}$.

As has been shown,(1) the steady state operating conditions for the laser may be obtained once $\mathcal{P}_n(t)$ is known as a function of the cavity field beyond the linear approximation.

Quantum Mechanical Liouville Equation

The time derivative of Eq. (11) along with the Schroedinger equation (10) lead to

$$\frac{\partial f_N}{\partial t} = \frac{i}{i\hbar} \int \frac{d^{6N}\lambda}{(2\pi)^{6N}} e^{-i\lambda\xi} \left[\mathcal{H}(\eta + \frac{\hbar}{2}\lambda) - \mathcal{H}(\eta - \frac{\hbar}{2}\lambda) \right] \Psi^*(\eta - \frac{\hbar}{2}\lambda, t)\, \Psi(\eta + \frac{\hbar}{2}\lambda, t)\,. \qquad (29)$$

The contribution to $\partial f_N/\partial t$ which is made by each of the four types of terms in $\mathcal{H}$ will now be considered separately.

The kinetic energy term given in Eq. (13) is readily shown(3) to contribute the convective term

$$\left.\frac{\partial f_N}{\partial t}\right|_{I} = -\frac{1}{m} \sum P_i \cdot \frac{\partial f_N}{\partial Q_i} \,. \qquad (30)$$

The contribution from the internal dynamics will merely be rewritten in the form

$$\left.\frac{\partial f_N}{\partial t}\right|_{II} = \frac{1}{i\hbar} \int \frac{d^{6N}\lambda}{(2\pi)^{6N}} \int d^{6N}\zeta\, e^{-i(\xi-\zeta)\lambda} \left[\sum_i (H_i + \sum_j W_{ij})_+ - \sum_i (H_i + \sum_j W_{ij})_- \right] f_N(\zeta, \eta, t), \quad (31)$$

where the subscripts $\pm$ refer to the displaced arguments of the type introduced in Eq. (29), and use has been made of the inverse Fourier

transform relation associated with Eq. (11). The rate of change of f_N resulting from the third term in $\mathcal{H}$ is

$$\left.\frac{\partial f_N}{\partial t}\right|_{III} = ie\sum_i E(Q_i,t)\cdot\int\frac{d^{6N}\lambda}{(2\pi)} e^{-i\lambda\cdot\zeta}T_i\,\Psi^*(\eta-\frac{\hbar}{2}\lambda,t)\,\Psi(\eta+\frac{\hbar}{2}\lambda,t)$$
$$= e\sum_i E(Q_i,t)\cdot\frac{\partial f_N}{\partial p_i}\;. \tag{32}$$

Finally, the part of the Hamiltonian containing the interaction due to translational motion leads to

$$\left.\frac{\partial f_N}{\partial t}\right|_{IV} = \frac{1}{i\hbar}\int\frac{d^{6N}\lambda}{(2\pi)^{6N}}\int d^{6N}\zeta\, e^{-i(\xi-\zeta)\cdot\lambda}\sum_{i<j}\Big[U_{ij}(|Q_i+\frac{\hbar}{2}T_i-Q_j-\frac{\hbar}{2}T_j|)$$
$$-U_{ij}(|Q_i-\frac{\hbar}{2}T_i-Q_j+\frac{\hbar}{2}T_j|)\Big]\,f_N(\zeta,\eta,t)\;. \tag{33}$$

The center of mass motion will be treated classically. This limit is readily obtained by expanding the integrand in Eq. (33) in powers of Planck's constant and retaining only the first nonvanishing term. One recovers the standard expression of kinetic theory,

$$\left.\frac{\partial f_N}{\partial t}\right|_{IV} = \sum_{i<j}\left(\frac{\partial U_{ij}}{\partial Q_i}\cdot\frac{\partial}{\partial P_i}-\frac{\partial U_{ij}}{\partial Q_j}\cdot\frac{\partial}{\partial P_j}\right)f_N(\zeta,\eta,t)\;. \tag{34}$$

As mentioned previously, f_N also changes due to an external pumping process which replenishes the excited states a and b. This change in f_N will be accounted for by including a phenomenological source term $\mathcal{Q}(\xi,\eta,t)$ in the quantum Liouville equation satisfied by f_N. The combination of Eqs. (30), (31), (32) and (34) and the source term yields the quantum Liouville equation in the form

$$\frac{\partial f_N}{\partial t}+\frac{1}{m}\sum P\cdot\frac{\partial f_N}{\partial Q}+\frac{1}{i\hbar}\int\frac{d^{6N}\lambda}{(2\pi)^{6N}}\int d^{6N}\zeta\, e^{-i(\xi-\zeta)\lambda}\Big[\sum(H_i+\sum_j W_{ij})_+$$
$$-\sum(H_i+\sum W_{ij})_-\Big]f_N(\zeta,\eta,t)-e\sum E(Q_n,t)\cdot\frac{\partial f_N}{\partial p_n} = \sum_{i<j}\left(\frac{\partial U_{ij}}{\partial Q_i}\cdot\frac{\partial}{\partial P_i}-\frac{\partial U_{ij}}{\partial Q_j}\cdot\frac{\partial}{\partial P_j}\right)f_N+\mathcal{Q}(\xi,\eta,t)\;. \tag{35}$$

Kinetic Equation Satisfied by the One-Particle Wigner Distribution Function

It has been shown previously that the physically interesting quantity, the macroscopic polarization density, may be expressed in terms of a one-particle Wigner distribution function. One obtains the equation determining this function by integrating Eq. (35) over the phase space of all particles except one. Employing the definitions given in Eqs. (19) and (20) this procedure yields

$$\frac{\partial f_1}{\partial t} + \frac{P_1}{m} \cdot \frac{\partial f_1}{\partial Q_1} + \sum_{n,m=a,b,c} (i\,\omega_{nm} + \gamma_{nm}) F_{1nm} f_{1mn} - e\,E(Q_1,t) \cdot \frac{\partial f_1}{\partial p}$$

$$= n^{(0)} \int d^6\xi_2\, d^6\eta_2\, \frac{\partial U_{12}}{\partial Q_1} \cdot \frac{\partial f_{12}}{\partial P_1} + \mathcal{Q}_1(\xi_1, \eta_1, t)\,, \tag{36}$$

where

$$\mathcal{Q}_1(\xi_1, \eta_1, t) = \int d^{6(N-1)}\xi\, d^{6(N-1)}\eta\, \mathcal{Q}(\xi, \eta, t)\,, \tag{37}$$

and

$$\begin{aligned} \gamma_{mn} &= \gamma_{nm} = (\gamma_m + \gamma_n)/2 \\ \omega_{mn} &= -\omega_{nm} = (E_m - E_n)/\hbar\,. \end{aligned} \tag{38}$$

It will prove useful to rewrite the term containing the external field by using the identity

$$e\,E(Q_1,t) \cdot \frac{\partial f_1}{\partial p_1} = -ie\,E(Q_1,t) \cdot \int \frac{d^6\lambda_1}{(2\pi)^6}\, e^{-i\lambda_1 \cdot \xi_1}\, \tau_1\, \psi_1^*\!\left(\eta_1 - \frac{\hbar}{2}\lambda_1, t\right) \psi\!\left(\eta_1 + \frac{\hbar}{2}\lambda_1, t\right), \tag{39}$$

where $d^6\lambda_1 = d^3T_1 d^3\tau_1$. In terms of the functions F_{1nm} introduced previously,

$$eE(Q_1,t)\cdot\frac{\partial f_1}{\partial p_1} = -ieE(Q_1,t)\cdot\int\frac{d^3\tau}{(2\pi)^3}e^{-i\tau_1\cdot P_1 \tau_1}\sum F_{1nm}\chi_m^*(q_1-\frac{\hbar}{2}\tau_1)\,\chi_n(q_1+\frac{\hbar}{2}\tau_1)\,. \qquad (40)$$

Modification of the Collision Term for Long Range Forces

The one term in Eq. (36) requiring further elucidation is the collision integral. Since the distribution function f_2 satisfies an equation containing integrals over f_3 which describes three particle effects etc., a means of closing this hierarchy must be introduced. Long-range forces may be treated by truncating Eq. (36) in a manner similar to that employed in plasma kinetic theory. Neglecting particle correlation entirely and writing

$$f_2(\xi_1,\xi_2,\eta_1,\eta_2,t) = f_1(\xi_1,\eta_1,t)\,f_1(\xi_2,\eta_2,t)\,, \qquad (41)$$

a pair of Vlasov-like equations may be derived for the distribution functions F_{aa} and F_{ab} introduced previously. It should be emphasized that no expansion in any plasma parameter is employed; spatial inhomogeneity in the assemblage of two-level systems is assumed to be due solely to the presence of the externally applied optical field. One finds that the introduction of Eq. (40) into Eq. (36) yields

$$\frac{\partial f_1}{\partial t} + \frac{P_1}{m}\cdot\frac{\partial f_1}{\partial Q_1} + \sum_{n,m=a,b,c}(i\omega_{nm}+\gamma_{nm})\,F_{nm}f_{mn} - eE\cdot\frac{\partial f_1}{\partial p_1} + e\mathcal{E}\cdot\frac{\partial f_1}{\partial P_1} = \mathcal{Q}(P_1,Q_1,t)\,, \qquad (42)$$

where

$$e\,\mathcal{E} = -n^{(0)}\frac{\partial}{\partial Q_1}\int d^3p_2 d^3Q_2 U_{12}\left[F_{aa}+F_{bb}+F_{cc}\right]. \qquad (43)$$

Introducing a potential through the definition

$$\mathcal{E} = -\frac{\partial\phi}{\partial Q_1} \qquad (44)$$

and employing the orthogonality properties of the f_{ij}, namely,

$$(2\pi\hbar)^3\int d^3p\,d^3q\, f_{ij}(p,q)\, f_{ab}(p,q) = \delta_{ja}\,\delta_{bi} \;, \tag{45}$$

one finds

$$\frac{\partial F_{aa}}{\partial t} + \frac{P_1}{m}\cdot\frac{\partial F_{aa}}{\partial Q_1} + \gamma_a F_{aa} - i\frac{E\cdot\wp_{ab}}{\hbar}(F_{ab}-F_{ba}) - e\frac{\partial\phi}{\partial Q_1}\cdot\frac{\partial F_{aa}}{\partial P_1} = \lambda_a \tag{46}$$

$$\frac{\partial F_{ab}}{\partial t} + \frac{P_1}{m}\cdot\frac{\partial F_{ab}}{\partial Q_1} + i\Omega_{ab}F_{ab} - i\frac{E\cdot\wp_{ab}}{\hbar}(F_{aa}-F_{bb}) - e\frac{\partial\phi}{\partial Q_1}\cdot\frac{\partial F_{ab}}{\partial P_1} = 0 \tag{47}$$

$$\frac{\partial F_{cc}}{\partial t} + \frac{P_1}{m}\cdot\frac{\partial F_{cc}}{\partial Q_1} - e\frac{\partial\phi}{\partial Q_1}\cdot\frac{\partial F_{cc}}{\partial P_1} = 0 \;, \tag{48}$$

where

$$\lambda_a(P_1,Q_1,t) \equiv (2\pi\hbar)^3\int d^3p_1\,d^3q_1\, f_{1ca}\,\mathcal{Q}_1(\xi_1,\eta_1,t) \;. \tag{49}$$

When these equations are expanded to third order in the optical field strength, it is found that the equations governing $F_{aa}^{(0)}$ and $F_{ab}^{(1)}$ are the same as those in the collisionless theory, namely,

$$\frac{\partial F_{aa}^{(0)}}{\partial t} + \frac{P}{m}\cdot\frac{\partial F_{aa}^{(0)}}{\partial Q} + \gamma_a F_{aa}^{(0)} = \lambda_a(P,Q,t) \tag{50}$$

$$\frac{\partial F_{ab}^{(1)}}{\partial t} + \frac{P}{m}\cdot\frac{\partial F_{ab}^{(1)}}{\partial Q} + i\Omega_{ab}F_{ab}^{(1)} = \frac{i}{\hbar}E\cdot\wp_{ab}\left[F_{aa}^{(0)} - F_{bb}^{(0)}\right] \tag{51}$$

$$\frac{\partial F_{aa}^{(2)}}{\partial t} + \frac{P}{m}\cdot\frac{\partial F_{aa}^{(2)}}{\partial Q} + \gamma_a F_{aa}^{(2)} = \frac{i}{\hbar}E\cdot\wp_{ab}\left[F_{ab}^{(1)} - F_{ba}^{(1)}\right] \tag{52}$$

$$\frac{\partial F_{ab}^{(3)}}{\partial t} + \frac{P}{m} \cdot \frac{\partial F_{ab}^{(3)}}{\partial Q} + i\Omega_{ab} F_{ab}^{(3)} = \frac{i}{\hbar} E \cdot \wp_{ab} \left[F_{aa}^{(2)} - F_{bb}^{(2)} \right], \tag{53}$$

where

$$\Omega_{ab} = \omega_{ab} - i\gamma_{ab} \; . \tag{54}$$

One also obtains

$$\frac{\partial F_{aa}^{(2)}}{\partial t} + \frac{P_l}{m} \cdot \frac{\partial F_{aa}^{(2)}}{\partial Q_l} + \gamma_a F_{aa}^{(2)} - e \frac{\partial \phi^{(2)}}{\partial Q_l} \cdot \frac{\partial F_{aa}^{(0)}}{\partial P_l} = \frac{iE \cdot \wp_{ab}}{\hbar} \left[F_{ab}^{(1)} - F_{ba}^{(1)} \right] \tag{55}$$

$$\frac{\partial F_{ab}^{(3)}}{\partial t} + \frac{P_l}{m} \cdot \frac{\partial F_{ab}^{(3)}}{\partial Q_l} + i\Omega_{ab} F_{ab}^{(3)} = \frac{iE \cdot \wp_{ab}}{\hbar} \left[F_{aa}^{(2)} - F_{bb}^{(2)} \right] + e \frac{\partial \phi^{(2)}}{\partial Q_l} \cdot \frac{\partial F_{ab}^{(1)}}{\partial P_l} \; . \tag{56}$$

It is seen that $F_{aa}^{(2)}$ is goverened by an inhomogeneous Vlasov equation and hence may be obtained through the use of standard integral transform techniques.

A Review of Collisionless Results

Before considering the effect of collisions, the collisionless results of Ref. (1) will first be recovered. From Eq. (50) one obtains

$$F_{aa}^{(00)}(P,Q,t) = \int_0^\infty d\tau \, e^{-\gamma_a \tau} \lambda_a (P, Q - P\tau/m, t - \tau) \, . \tag{57}$$

Assuming that

$$\lambda_a (P,Q,t) = \Lambda_a \Phi^{(0)}(P) \; , \tag{58}$$

where Λ_a is constant or varies slowly compared to the other length

and time scales of interest in the problem, one may write

$$F_{aa}^{(00)}(P,Q,t) = \Lambda_a \Phi^{(0)}(P)/\gamma_a \,. \tag{59}$$

For use in Eq. (51), it is useful to write

$$F_{aa}^{(00)} - F_{bb}^{(00)} = \Phi^{(0)} N/n^{(0)} \tag{60}$$

where

$$N \equiv \frac{\Lambda_a}{\gamma_a} - \frac{\Lambda_b}{\gamma_b} \tag{61}$$

and is referred to[1] as the "excitation density". The equation obtained by interchanging a and b in Eq. (51) is

$$F_{ab}^{(0)}(P,Q,t) - F_{ba}^{(0)}(P,Q,t) = i\delta (N/n^{(0)})\, \Phi^{(0)}(P) \int_0^\infty d\tau \,(e^{-i\Omega_{ab}\tau} + e^{-i\Omega_{ba}\tau}) \times U_n(Q - P\tau/m) A_n(t-\tau) \,, \tag{62}$$

where the electromagnetic field has been written

$$E(Q,t) = eE_n A_n(t) U_n(Q) \tag{63}$$

and

$$\delta = E_n \wp/\hbar \,. \tag{64}$$

Obtaining a similar solution for $F_{aa}^{(20)} - F_{bb}^{(20)}$ from Eq. (52) and substituting the result into Eq. (53) leads to

$$\begin{aligned} F_{ab}^{(30)}(P,Q,t) = {} & (i\delta)^3 (N/n^{(0)})\, \Phi^{(0)}(P) \sum_{c=a,b} \int_0^\infty d\tau\, d\tau' d\tau'' e^{-i\Omega_{ab}\tau - \gamma_c' \tau'} \\ & \times (e^{-i\Omega_{ab}\tau''} + e^{-i\Omega_{ba}\tau''}) A_n(t-\tau) A_n(t-\tau-\tau') A_n(t-\tau-\tau'-\tau'') \\ & \times U_n(Q - P\tau/m) U_n\left[Q - (P\tau + P\tau')/m\right] U_n\left[Q - (P\tau + P\tau' + P\tau'')/m\right]. \end{aligned} \tag{65}$$

From Eq. (28), the first-order contribution to the polarization is

$$\mathcal{P}_n^{(10)}(t) = -n^{(0)} \mathscr{P} \frac{2}{L} \int d^3P \int_0^L dZ U_n(Z) F_{ab}^{(10)}(P,Q,t) + \text{c.c} \quad . \tag{66}$$

Introducing $F_{ab}^{(10)}$, carrying out the spatial integration and neglecting the second harmonic contribution yields

$$\begin{aligned}\mathcal{P}_n^{(10)}(t) &= -i8\,\mathscr{P}N \int d^3P\Phi^{(0)}(P) \int_0^\infty d\tau e^{-i\Omega_{ab}\tau} A_n(t-\tau) \cos k_n P\tau/m + \text{c.c.} \\ &= -i8\,\mathscr{P}N \int_0^\infty d\tau e^{-\Omega_0^2\tau^2 - i\Omega_{ab}\tau} A\,(t-\tau) + \text{c.c.}\end{aligned} \tag{67}$$

where

$$\Omega_0 = k_n p_0/2m \quad . \tag{68}$$

Setting

$$A_n(t) = \cos(\omega_n t + \phi) \tag{69}$$

and using the rotating wave approximation(1) one obtains

$$\mathcal{P}_n^{(10)}(t) = -\frac{8\mathscr{P}N}{2k_n\nu_0} e^{-i(\omega_n t+\phi)} Z\left(\frac{\omega_n - \Omega_{ab}}{k_n\nu_0}\right) + \text{c.c.} \tag{70}$$

where

$$Z(\zeta) \equiv i\int_0^\infty dt\, e^{i\zeta t - t^2/4} = Z_r + iZ_i \quad . \tag{71}$$

In Ref. (1), it is shown that the steady state operating conditions follow from a knowledge of that part of $\mathcal{P}_n$ that varies as $\sin(\omega_n t+\phi)$.

Writing

$$\mathcal{P}_n(t) = C_n \cos(\omega_n t + \phi) + S_n \sin(\omega_n t + \phi) \tag{72}$$

there results

$$S_n^{(10)} = -\frac{\wp^2 N E_n}{\hbar k_n v_o} Z_i\left(\frac{\omega_n - \Omega_{ob}}{k_n v_o}\right). \tag{73}$$

The third-order contribution is obtained from Eqs. (28) and (65) by a similar but somewhat more lengthy procedure. The spatial integration is first carried out by employing the formula

$$\begin{aligned} &\frac{2}{L}\int_0^L dZ\, U_n(Z) U_n(Z-a_1) U_n(Z-a_2) U_n(Z-a_3) \\ &= \frac{1}{4}\left[\cos(a_1+a_2-a_3) + \cos(a_2+a_3-a_1) + \cos(a_3+a_1-a_2)\right] . \end{aligned} \tag{74}$$

In the present case

$$\begin{aligned} a_1 + a_2 - a_3 &= k_n \cdot P(\tau-\tau'')/m \\ a_2 + a_3 - a_1 &= k_n \cdot P(\tau+2\tau'+\tau'')/m \\ a_3 + a_1 - a_2 &= k_n \cdot P(\tau+\tau'')/m \quad . \end{aligned} \tag{75}$$

As has been pointed out,[1] the second and third of these terms give contributions with higher inverse powers of $k_n v_o$ and hence in the "extreme Doppler limit" may be neglected. Carrying

out the momentum integration then yields

$$\rho_n^{(30)}(t) = -\frac{1}{4}\wp(i\delta)^3 N \sum_{c=a,b} \int_0^\infty d\tau\, d\tau'\, d\tau''\, e^{-i\Omega_{ab}\tau - \gamma_c'\tau' - \Omega_n(\tau-\tau'')^2}$$

$$\times (e^{-i\Omega_{ab}\tau''} + e^{-i\Omega_{ba}\tau''}) A_n(t-\tau)\, A_n(t-\tau-\tau')\, A_n(t-\tau-\tau'-\tau'') + \text{c.c.}\,. \tag{76}$$

When the three time factors are written in exponential form, it is found that only the terms $(1/8)e^{-i(\omega_n t+\varphi)+i\omega_n(\tau+\tau'')}$ and $(1/8)e^{-i(\omega_n t+\varphi)+i\omega_n(\tau-\tau'')}$ give rise to denominators that will have a resonance when the various time integrations in Eq. (76) are performed. Retaining only these terms, Eq. (76) becomes

$$\rho_n^{(30)}(t) = -\frac{1}{32}\wp(i\delta)^3 N e^{-i(\omega_n t+\varphi)} \sum_{c=a,b} \frac{1}{\gamma_c} \int_0^\infty d\tau\, d\tau' d\tau''\, e^{-\Omega_o^2(\tau-\tau'')^2}$$

$$\times e^{-i(\Omega_{ab}-\omega_n)\tau}\left[e^{-i(\Omega_{ab}-\omega_n)\tau''} + e^{-i(\Omega_{ba}+\omega_n)\tau''}\right] + \text{c.c.} \tag{77}$$

$$= -\frac{1}{32}\wp(i\delta)^3 N e^{-i(\omega_n t+\varphi)}\left(\frac{1}{\gamma_a}+\frac{1}{\gamma_b}\right)\left[I(\Omega_o,\omega_1,\omega_1) + I(\Omega_o,\omega_1,-\omega_1^*)\right] + \text{c.c.}\,,$$

where

$$\omega_1 = \omega_n - \Omega_{ab}\,, \tag{78}$$

and

$$I(\Omega_o,\omega,\omega') \equiv \int_0^\infty d\tau d\tau' e^{-\Omega_o^2(\tau-\tau')^2 + i\omega\tau + i\omega'\tau'}. \tag{79}$$

One integration in this expression is readily performed after introducing the transformation

$$x = \tau_1 + \tau_2 \qquad y = \tau_1 - \tau_2 . \tag{80}$$

The remaining integrals are expressible in terms of the function introduced in Eq. (71). One finds

$$I(\Omega_0, \omega, \omega') = \left[Z\left(\frac{\omega}{2\Omega_0}\right) + Z\left(\frac{\omega'}{2\Omega_0}\right)\right] / \left[2\Omega_0(\omega+\omega')\right]. \tag{81}$$

Since Z(-z*) = -Z*(z), it follows that

$$I(\Omega_0, \omega_1, \omega_1) = (k_n v_0 \omega_1)^{-1} Z(\omega_1/2\Omega_0), \tag{82}$$

and

$$I(\Omega_0, \omega_1; \omega_1) = (k_n v_0 \mathrm{Im}\,\omega_1)^{-1} Z_i(\omega_i/2\Omega_0). \tag{83}$$

In terms of these results, the third-order polarization becomes

$$\mathcal{P}_n^{(30)}(t) = -\frac{\wp(i8)N}{32 k_n v_0}\left(\frac{1}{\gamma_a} - \frac{1}{\gamma_b}\right) e^{-i(\omega_n t + \phi)}\left[\omega_1^{-1} Z(\omega_1/k_n v_0) + (\mathrm{Im}\,\omega_1)^{-1} Z_i(\omega_1/k_n v_0)\right]. \tag{84}$$

From which one finds

$$S_n^{(30)} = \frac{\wp^4 E^3 N}{8\hbar^3 \gamma_a \gamma_b k_n v_0} Z_i(\omega_1/k_n v_0)\left[1 + \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_{ab})\right], \tag{85}$$

where

$$\mathcal{L}(\omega) \equiv \frac{1}{\omega^2 + \gamma_{ab}^2} . \tag{86}$$

Now the operating condition for a laser is obtained from the steady state solution of the equation[1]

$$\dot{E}_n = \alpha_n E_n - \beta_n E_n^3 \quad , \qquad (87)$$

where

$$\alpha_n = \frac{1}{2}\left(\frac{\omega_n}{Q_n}\right) - \frac{1}{2}\left(\frac{\omega_n}{\epsilon_o}\right)\frac{S_n^{(1)}}{E_n} \quad , \qquad (88)$$

$$\beta_n = \frac{1}{2}\left(\frac{\omega_n}{\epsilon_o}\right)\frac{S_n^{(3)}}{E_n^3} \quad , \qquad (89)$$

while Q_n is the Q of the n^{th} cavity mode and ϵ_o is the permittivity of free space.

The threshold excitation density for laser operation is obtained by setting $\alpha_n = 0$ in Eq. (88). Neglecting effects that are $0(\epsilon)$, one finds that the threshold density at resonance ($\omega_n = \omega_{ab}$) is

$$N_T = \frac{\epsilon_o \hbar k_n v_o}{Q_n \wp^2 \sqrt{\pi}} \quad . \qquad (90)$$

Setting $\dot{E}_n = 0$ in Eq. (87) and introducing Eqs. (73), (85) and (90), the equilibrium field amplitude is given by

$$\frac{\wp^2 E_n^2}{8\hbar^2 \gamma_a \gamma_b} = \frac{1 - \eta^{-1} e^{(\omega_n - \omega_{ab})^2/(k_n v_o)^2}}{1 + \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_{ab})} \quad , \qquad (91)$$

where

$$\eta = N/N_T \quad . \qquad (92)$$

In obtaining this result the approximation

$$Z_i(x) = \pi^{1/2} e^{-x^2} \quad , \tag{93}$$

has been used in the numerator to yield the Doppler profile. In the denominator this exponential has been replaced by unity since the dip in the response curve occurs in the region $|\omega_n - \omega_{ab}| \leq \gamma_{ab}$ for which the exponential is essentially constant in the extreme Doppler limit.

Solution of Vlasov Equation for $F_{aa}^{(2)}$

The solution of Eq. (55) may be carried out by introducing the transforms

$$F_{aa}^{(2)}(P,Q,t) = \sum_\ell e^{ik_\ell \cdot Q} \int \frac{dS}{2\pi i} e^{st} \mathfrak{F}_{a\ell}^{(2)}(P,s) \quad , \tag{94}$$

$$\phi^{(2)}(Q,t) = \sum_\ell e^{ik_\ell \cdot Q} \int \frac{ds}{2\pi i} e^{st} \phi_\ell^{(2)}(s) \quad , \tag{95}$$

$$h_a^{(2)}(P,Q,t) = \sum_\ell e^{ik_\ell \cdot Q} \int \frac{ds}{2\pi i} e^{st} H_{a\ell}^{(2)}(P,s) \quad , \tag{96}$$

where $h_a^{(2)}(P,Q,t)$ is the inhomogeneous term in Eq. (55). The transformed version of Eq. (55) is

$$(s+\gamma_a + ik_\ell \cdot P_1/m)\mathfrak{F}_{a\ell}^{(2)} + iek_\ell \cdot \frac{\partial F_{aa}^{(0)}}{\partial P_1} \phi_\ell^{(2)} = H_{a\ell}^{(2)} \quad . \tag{97}$$

From Eq. (43) it follows that

$$\phi_\ell^{(2)}(s) = \frac{4\pi n_n^{(0)} e}{k_\ell^{\ 2}} \int d^3P \left[\mathcal{F}_{a\ell}^{(2)} + \mathcal{F}_{b\ell}^{(2)} + \mathcal{F}_{c\ell}^{(2)} \right] \quad . \tag{98}$$

Equation (97), along with a similar equation for $F_{b\ell}^{(2)}$ plus a homogeneous equation for $F_{c\ell}^{(2)}$ (since $\wp_{ac} = \wp_{bc} = 0$) may now be solved simultaneously. One finds

$$\mathcal{F}_{a\ell}^{(2)}(P,s) = \mathcal{F}_{a\ell}^{(20)}(P,s) + \frac{4\pi n^{(0)} e^2}{k_\ell^{\ 2}} \frac{n_\ell^{(20)}(s)}{n^{(0)}\epsilon_\ell(s)} \frac{i k_\ell \cdot \frac{\partial \Phi^{(0)}}{\partial P}}{s + \gamma_a + i k_\ell \cdot P/m} \quad , \tag{99}$$

where

$$\mathcal{F}_{a\ell}^{(20)}(P,s) = \frac{H_{a\ell}(P,s)}{s + \gamma_a + i k_\ell \cdot P/m} \quad , \tag{100}$$

and

$$n_\ell^{(20)}(s) = \int d^3P \left[\mathcal{F}_{a\ell}^{(20)} + \mathcal{F}_{a\ell}^{(20)} + \mathcal{F}_{a\ell}^{(20)} \right] \quad . \tag{101}$$

Also, $\epsilon_\ell(s)$ is defined by

$$\epsilon_\ell(s) = \epsilon_{a\ell}(s) + \epsilon_{b\ell}(s) + \epsilon_{c\ell}(s) - 2 \quad , \tag{102}$$

where

$$\epsilon_{n\ell}(s) = 1 - \frac{4\pi n_n^{(0)} e^2}{k_\ell^{\ 2}} \int d^3P \frac{i k_\ell \cdot \frac{\partial \Phi^{(0)}}{\partial P}}{s + \gamma_n + i k_\ell \cdot P/m} \quad , \qquad n = a, b, c \quad . \tag{103}$$

Third-Order Polarization for the Ion Laser

The third-order polarization follows in the usual way from Eq. (28),

$$\mathcal{P}_n^{(3)}(t) = -n^{(0)}\wp_{ab}\frac{2}{L}\int_0^L dZ\,U_n(Z)\int d^3P\,F_{ab}^{(3)}(P,Q,t) + \text{c.c.} \quad . \tag{104}$$

From Eq. (56) the appropriate form for $F_{ab}^{(3)}$ is found to be

$$F_{ab}^{(3)}(P,Q,t) = (iE_o\cdot\wp_{ab}/\hbar)\int_0^\infty d\tau\, e^{-i\Omega_{ab}\tau}A_n(t-\tau)\,U_n(Q-P\tau/m)\Big[F_{aa}^{(2)}(P,Q-P\tau/m,t-\tau)$$
$$-F_{bb}^{(2)}(P,Q-P\tau/m,t-\tau)\Big] + e\int_0^\infty d\tau\, e^{-i\Omega_{ab}\tau}\mathcal{E}^{(2)}(Q-P\tau/m,t-\tau)\cdot\frac{\partial}{\partial P}F_{ab}^{(1)}(P,Q-P\tau/m,t-\tau) \quad . \tag{105}$$

The spatial integration required in the evaluation of the polarization density is

$$\frac{2}{L}\int_0^L dZ\,U_n(Z)U_n(Z-V\tau)e^{ik_\ell Z} = \frac{1}{2}\left(2\delta_{\ell,0}\cos k_nV\tau - \delta_{\ell,2n}e^{ik_\ell V\tau} - \delta_{\ell,-2n}e^{-ik_\ell V\tau}\right) \quad . \tag{106}$$

It is found that the dominant correction to the polarization comes from the $\ell = 0$ term in $F_{aa}^{(2)}-F_{bb}^{(2)}$. All other terms yield contributions that are higher order in $(k_nv_o)^{-1}$. Hence we set

$$\Delta\mathcal{P}^{(3)} = \mathcal{P}_n^{(3)} - \mathcal{P}_n^{(30)} = -\wp_{ab}(iE_o\cdot\wp_{ab}/\hbar)\int_0^\infty d\tau\, e^{-i\Omega_{ab}\tau}A_n(t-\tau)\int d^3P\int\frac{ds}{2\pi i}\,e^{s(t-\tau)}$$
$$\times\, e^{-ik_\ell\cdot P\tau/m}\,\frac{mn_\ell^{(20)}(s)\,\omega_p^2}{k_\ell^2(s^2+\omega_p^2)}\left(\frac{1}{s+\gamma_a+ik_\ell\cdot P/m} - \frac{1}{s+\gamma_b+ik_\ell\cdot P/m}\right)\delta_{\ell,0}\cos k_n\cdot P\tau/m \quad , \tag{107}$$

where $\omega_p^2 = 4\pi n^{(0)}e^2/m$.

The $\ell = 0$ form of the dielectric constant, namely

$$\epsilon_o(s) = 1 + s^2/\omega_p^2 \quad , \tag{108}$$

has already been introduced in Eq. (107) since it is insensitive to the limiting process required in evaluating this expression at $k_\ell = 0$.

Performing the s-inversion first, the correction to the third-order polarization may be written

$$\Delta \mathcal{P}^{(3)} = W_a + W_t + c.c. \quad , \tag{109}$$

where

$$W_a = -\mathcal{E}_{ab}\left(\frac{iE_o \cdot \mathcal{E}_{ab}}{\hbar}\right)\frac{m\omega_p}{2i}\frac{\omega_{pa}^2}{k_\ell^2}\int_0^\infty d\tau\, e^{-i\Omega_{ab}\tau} A_n(t-\tau)\int d^3P\, e^{-ik_\ell\cdot P\tau/m}\cos k_n\cdot P\tau/m$$
$$\times \frac{ik_\ell \cdot \dfrac{\partial \Phi^{(0)}}{\partial P}}{\gamma_a + i\omega_p + ik_\ell \cdot P/m}\int_0^\infty d\tau'\, N_\ell^{(20)}(t-\tau-\tau')\left[e^{i\omega_p\tau'} - e^{-(\gamma_a + i\omega_p)\tau'}\right] \tag{110}$$
$$+ \text{(similar term with } \omega_p \rightarrow -\omega_p) \quad ,$$

and $\omega_{pa}^2 = 4\pi n_a^{(0)} e^2/m$.

The second order density variation to be used in Eq. (110) is readily obtained from Eqs. (50) - (52). The result is

$$N_o^{(20)}(t) = -\frac{1}{2}N(iE_o \cdot \mathcal{E}_{ab}/\hbar)^2\int_0^\infty d\tau\, d\tau'\, e^{-\Omega_e^2\tau'^2}(e^{-\gamma_a\tau} - e^{-\gamma_b\tau})$$
$$\times (e^{-i\Omega_{ab}\tau'} + e^{-i\Omega_{ba}\tau'})A_n(t-\tau)A_n(t-\tau-\tau') \quad . \tag{111}$$

Integration over momentum in Eq. (110) requires the evaluation of the integral

$$I_a(t_1, t_2) = \int d^3P \frac{ik_\ell \cdot \dfrac{\partial \Phi^{(0)}}{\partial P}\, e^{-ik_\ell \cdot Pt_2/m}\cos k_n \cdot Pt_1/m}{\gamma_a + i\omega_p + ik_\ell \cdot P/m} \quad . \tag{112}$$

Expanding the denominator and only retaining terms $O(k_\ell^{\,2})$ yields

$$I_a(t_1,t_2) = -k_\ell^2 e^{-(\Omega_o t_1)^2}\left[1-2(\Omega_o t_1)^2\right]\left[1+t_2(\gamma_a+i\omega_p)\right]/(\gamma_a+i\omega_p)^2 \quad . \tag{113}$$

When Eqs. (110) and (111) are combined and the three time factors $A_n(t)$ are written in exponential form, it is again found that only two terms give rise to resonant contributions in subsequent time integrations. Hence Eq. (110) may be written

$$\begin{aligned} W_a = {} & \frac{1}{32} N\omega_p\omega_{pa}^2 \wp_{ab}(E_o\cdot \wp_{ab}/\hbar)\, e^{-i(\omega_n t+\phi)}\int_0^\infty d\tau\, e^{-\Omega_o^2\tau^2 - i(\Omega_{ab}-\omega_n)\tau}\left[1-2(\Omega_o\tau)^2\right] \\ & \times(\gamma_a+i\omega_p)^{-2}\int_0^\infty d\tau'\left\{e^{i\omega_p\tau'}\left[1+\tau(\gamma_a+i\omega_p)\right]-e^{-\gamma_a\tau'}\left[1+(\tau+\tau')(\gamma_a+i\omega_p)\right]\right\} \\ & \times\int_0^\infty d\tau'' d\tau''' e^{-\Omega_o^2\tau'''^2}\left(e^{-\gamma_a\tau''}-e^{-\gamma_b\tau''}\right)\left[e^{-i(\Omega_{ab}-\omega_n)\tau'''}+e^{-i(\Omega_{ba}+\omega_n)\tau'''}\right] \end{aligned} \tag{114}$$

$$+ \text{ (similar term with } \omega_p \rightarrow -\omega_p) \quad .$$

After performing the simple integrations of τ' and τ'', the τ''' integration is given in terms of the z function introduced in Eq. (71) and the τ integration requires

$$\begin{aligned} &\int_0^\infty d\zeta\, e^{-\zeta^2+i\epsilon\zeta}(1-2\zeta^2) = -\frac{i}{2}\epsilon + O(\epsilon^2) \\ &\int_0^\infty d\zeta\, e^{-\zeta^2+i\epsilon\zeta}\,\zeta(1-2\zeta^2) = -\frac{1}{2} + O(\epsilon) \quad . \end{aligned} \tag{115}$$

One finally obtains

$$W_a = -\frac{i N\omega_{pa}^2 \wp_{ab}}{4\gamma_a(\,_n v_o)^3}\left(\frac{E_o\wp_{ab}}{\hbar}\right)^3\left(\frac{1}{\gamma_a}-\frac{1}{\gamma_b}\right)\mathrm{Im}Z\left(\frac{\omega_n-\Omega_{ab}}{k_n v_o}\right)\left[1+i\left(\frac{\omega_n-\Omega_{ab}}{\gamma_a}\right)\right] \quad . \tag{116}$$

The resulting change in the polarization density may now be obtained

from Eq. (109) The associated in quadrature component is found to be

$$\Delta S_n^{(3)} = \frac{\wp_{ab} N}{4\gamma_a \gamma_b k_n v_0}\left(\frac{E_0 \wp_{ab}}{\hbar}\right)^3 \mathrm{Im}Z\left(\frac{\omega_n - \Omega_{ab}}{k_n v_0}\right)\left[\frac{1}{(\gamma_a k_n L_a)^2}+\frac{1}{(\gamma_b k_n L_b)^2}\right] , \qquad (117)$$

where $L_a^2 = kT/4\pi\, n_a^{(0)} e^2$.

Finally, the equilibrium field amplitude is given by

$$\frac{\wp_{ab}^2 E_n^2}{8\hbar \gamma_a \gamma_b} = \frac{1 - \eta^{-1} e^{(\omega_n - \Omega_{ab})^2/(k_n v_0)^2}}{1 + \gamma_{ab}^2 \mathcal{L}(\omega_n - \omega_{ab}) + 2(\gamma_a - \gamma_b)^2\left[(\gamma_a k_n L_a)^{-2} + (\gamma_b k_n L_b)^{-2}\right]} . \qquad (118)$$

Comparison of Eq. (118) with the corresponding result of Ref.(1)shows that the effect of long range forces is contained in the last term in the denominator of Eq. (118). This term is enhanced by increasing the wavelength of the laser radiation and by decreasing the Debye length associated with the particles in states a and b. Since these Debye lengths turn out to be on the order of 10^{-3} cm while the optical wavelengths are 10^{-5} - 10^{-4} cm, it is seen that the correction term would be quite small were it not for the fact that the transition rate γ_b of the lower level may be as much as $10\gamma_a$.

As a numerical example, consider 1μ radiation in a plasma having a temperature of 2000°K and $n_a^{(0)} \sim n_b^{(0)} \sim 5 \times 10^{11}$ cm^{-3}. Then $L_a^2 = L_b^2 = 2 \times 10^{-7}$ cm^2. Assuming that $\gamma_b = 10\gamma_a$, the correction term is 0.2. The response of the laser at frequencies above ω_{ab} both with and without this correction is shown in Fig. 1. The complete response curve is, of course, symmetric about the frequency ω_{ab}. Although the shape of the response curve remains unchanged when long range forces are included, a superposition of the two curves in Fig. 1 shows that the effect of such forces is to fill in the center of the curve. Also, the requirement of positive curvature at the center of the response curve yields as a criterion for the existence of a dip,

$$\eta > 1 + \frac{2\gamma_{ab}^2}{\Omega_0^2}\left(1 + \frac{c}{2}\right) . \qquad (119)$$

where

$$\frac{c}{2} = (\gamma_a - \gamma_b)^2 \left[(\gamma_a k_n L_a)^{-2} + (\gamma_b k_n L_b)^{-2} \right] \quad . \tag{120}$$

Finally, Eq. (118) shows that the correction term vanishes if $\gamma_a = \gamma_b$. This is clear on physical grounds since in the laser theory one finds that $F_{aa}^{(2)} = -F_{bb}^{(2)}$ when γ_a and γ_b are equal. These two terms then have identical spatial dependences that are 180° out of phase with each other. Since the net force acting on a given two-level system is due to the inhomogeneous distribution of other charged particles, it is clear that for $\gamma_a = \gamma_b$ the distribution of particles on state a plus those in state b gives rise to a completely homogeneous spatial distribution of such particles and no net force is felt by an individual two-level system.

MEASUREMENTS OF ARGON ION LASER LINE PROFILE

Introduction

In considering the output power of laser oscillators, it has been shown that the saturation characteristics of the optical medium depend upon the broadened line width of the medium and the frequency separation between modes.[4] A representation of this mechanism is most easily described as a hole being "burned" in the Doppler-broadened gain curve of a gas laser. The "hole burning" refers to the saturation of the atoms in the vicinity of the frequency of the saturating radiation.[5] At low power levels the distortions in the "hole width" are due to the broadening caused by collisions in a non-ionized plasma[6] and by the long range forces in an ionized plasma[7] (see the earlier section of this report).

Experimental Arrangements and Preliminary Results

For a cavity of very short length, the spacing of the resonances may exceed the Doppler-broadened line width and the laser will oscillate in a single mode. One expects oscillation at a single frequency to cause gain saturation of the medium only in the vicinity of this frequency. Two holes are "burned" in the gain curve due to the standing wave character of the optical fields in the cavity. If the length of the cavity is tuned through the center of the Doppler profile, a "dip" is observed which is due to the fact that fewer atoms are contributing to the power output of the laser. For the specific case of the ionized argon gas laser, the gain profile is so wide that a short single mode cavity could not be conveniently constructed and therefore the "Lamb Dip" could not be displayed by tuning the c/2L frequency through line center. If the output intensity of the argon laser was extremely stable (which it is not), it could be operated very close to threshold in one mode and the "Lamb Dip" could possibly be exhibited; however, no range of the excitation parameter would

exist.

An alternate solution which has been successfully applied during the reporting period is to use a three mirror, variable reflectivity system at one end of the resonator.[8] A /2 motion of this system will allow the two cavity system to pass through alignment and will provide a resonance for one of the modes of the original cavity. By linearly driving one of the mirrors of the variable reflectivity cavity the resonances may be scanned across the whole Doppler-broadened line and will exhibit the "Lamb Dip" as the particular oscillating modes are scanned in time. The "Lamb Dip" may now be plotted as a function of excitation and the area of the "hole" represents the relation power to be achieved by a single mode since the "hole" area represents the homogeneously broadened line width.

The experimental arrangement is shown pictorially in Fig. 2 and schematically in Fig. 3a. A d.c. Ar^+ discharge is contained in a quartz capillary which is 60 cm in length and has a 2 mm ID. The excitation can be varied from 3 to 10 amperes. An intracavity prism is incorporated into the design to enable the various atomic lines to be selected. The three mirror, variable reflectivity system is located at one end of the cavity and has a multiwafer piezoelectric drive associated with mirror M, which provides the scan of the short cavity relative to the long cavity.

The system has been designed, constructed and made operative. Preliminary data, as shown in Fig. 3b, has illustrated the "Lamb Dip", and demonstrated the suitability of the technique for a detailed investigation of the shape of the hole burned in the Doppler gain curve. Detailed results of the experimental portions of this program and attempts to interpret the experiments in terms of this theory will be reported in a later report.

SIX-MONTH STATUS EVALUATION

Some aspects of the theory of a gas laser recently developed by W. E. Lamb, Jr. have been recast in a form which more fully displays the role played by the particle dynamics. The Wigner distribution function has been used to derive kinetic equations which govern the external center of mass motion of the two-level systems as well as their internal dynamics. The effect of long range forces has been discussed by treating the collision integral in a manner similar to that employed in plasma kinetic theory. A modification in the criterion for the existence of a dip in the output has been obtained. It has also been shown that effects due to long range forces are most noticeable at long optical wavelengths and when there is a large difference between the lifetimes of the two laser levels.

The experimental system for measuring the line profile of a d.c. excited argon laser has been designed, constructed, and made operative. Preliminary data on the "Lamb Dip" has been obtained as a function of pressure and excitation. Detailed results of the experimental portions of this program and attempts to interpret the experiments in terms of this theory will be reported in a later report.

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LIST OF FIGURES

F-920479-2

FIG. 1

CALCULATED LAMB DIP LINE PROFILES

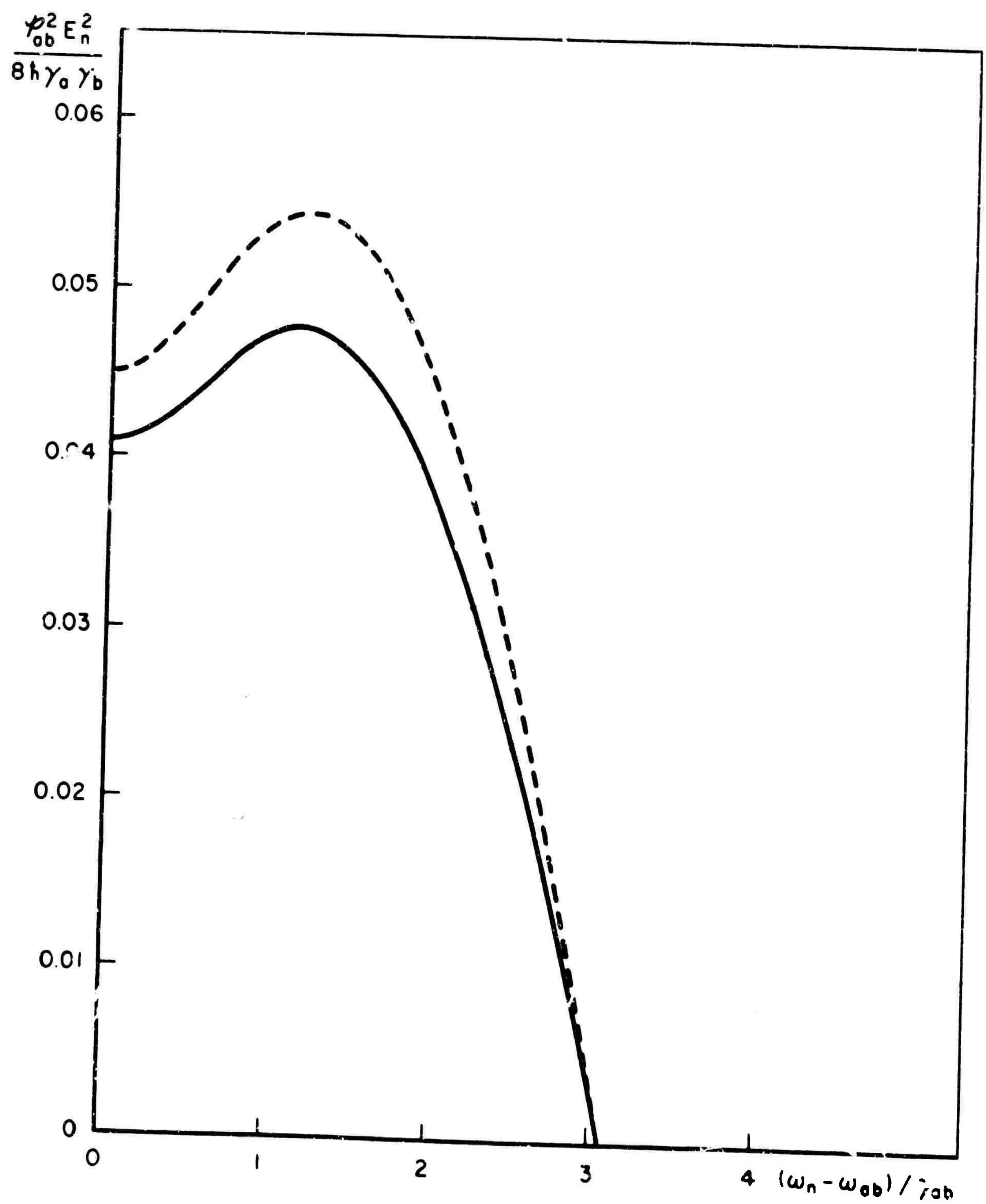

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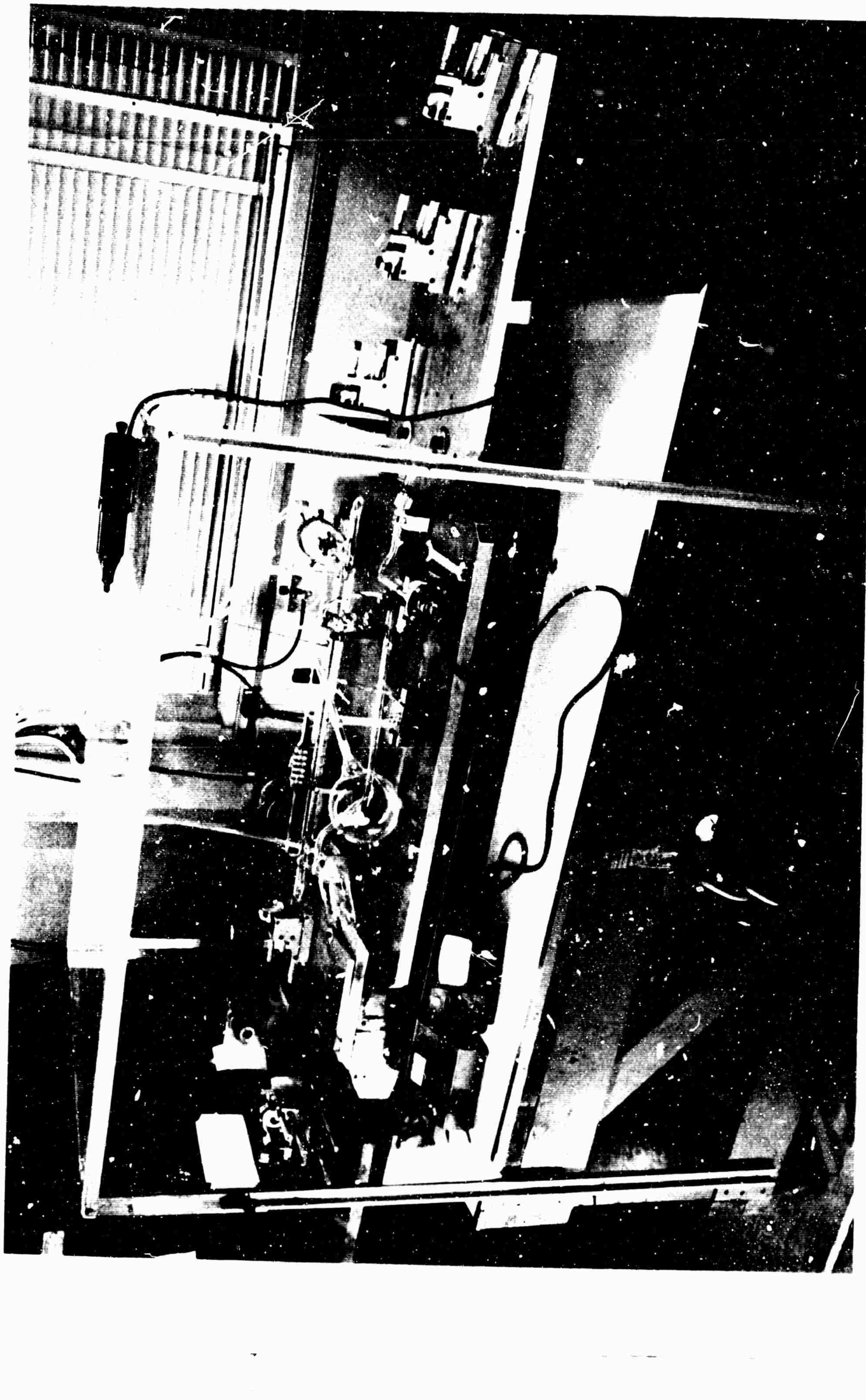
FIG. 2

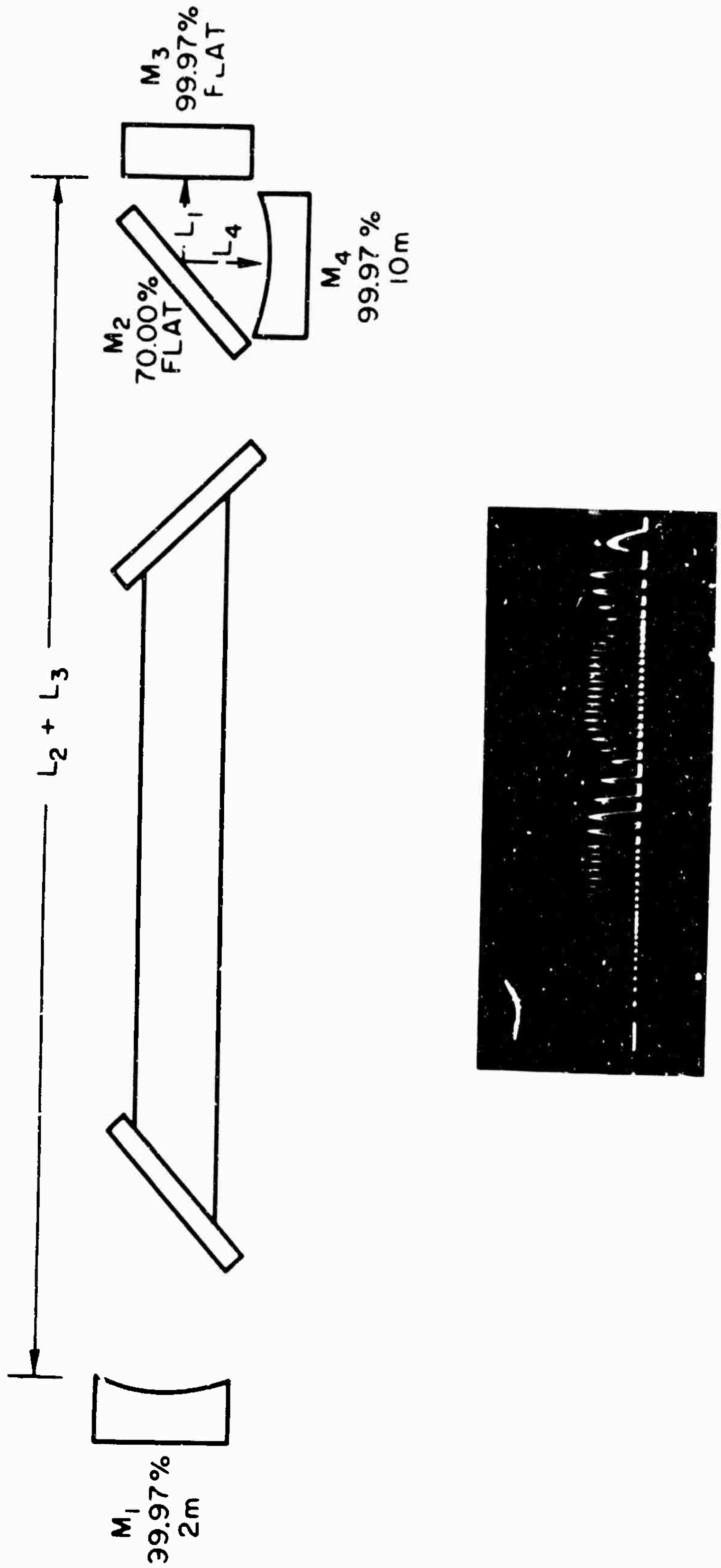


EXPERIMENTAL APPARATUS FOR LASER LINE PROFILES

F-320479-2

FIG. 3

SINGLE MODE OPERATION OF ARGON ION LASER



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13. ABSTRACT

Some aspects of the theory of a gas laser recently developed by W. E. Lamb, Jr. are recast in a form which more fully displays the role played by the particle dynamics. The Wigner distribution function is used to derive kinetic equations which govern the external center of mass motion of the two-level system as well as their internal dynamics. The effect of long range forces is discussed by treating the collision integral in a manner similar to that employed in plasma kinetic theory. A modification in the criterion for the existence of a dip in the output is obtained. It is also shown that effects due to long range forces are most noticeable at long optical wavelengths and when there is a large difference between the lifetimes of the two laser levels.

The experimental system for measuring the line profile of a dc excited argon laser has been designed, constructed, and made operative. Preliminary data on the "Lamb Dip" of an argon ion laser has been obtained as a function of pressure and excitation. The results of the experimental portion of this program and attempts to interpret the experiments in terms of this theory will be reported in later reports.

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